Rotational Motion

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi$$

$$+V\psi=E\psi(r,\theta,\phi) \quad \ldots \quad (1)$$

Particle on a ring: r is fixed; $\theta=\pi/2$; $V=0$

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\psi=E\psi(\phi) \quad \ldots \quad (2)$$

$$\frac{\partial^2\psi}{\partial\phi^2}=-\underbrace{\frac{2IE}{\hbar^2}}_{\Downarrow\, m_l^2}\psi(\phi) \quad \ldots \quad (3)$$

$$\frac{\partial^2\psi}{\partial\phi^2}=-m_l^2\psi(\phi)$$

$$\Downarrow$$

$$\Phi(\phi)=a_{m_l}e^{im_l\phi} \quad \ldots \quad (4)$$

$$\Phi(\phi)=\frac{1}{\sqrt{2\pi}}e^{im_l\phi} \quad \ldots \quad (5)$$

$$m_l=0,\pm1,\pm2,\pm3\ldots$$

$$\hat{l}_z=-i\hbar\frac{\partial}{\partial\phi} \quad \ldots \quad (6)$$

Particle on a sphere

r is fixed ; $V = 0$

$\Downarrow$

RIGID ROTOR problem

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\Psi = E\Psi(\theta,\phi) \quad \cdots (1)$$

$$-\frac{\hbar^2}{2mr^2}\underbrace{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]}_{\Lambda^2}\Psi = E\Psi(\theta,\phi) \quad \cdots (2)$$

$$-\frac{\hbar^2}{2mr^2}\Lambda^2\Psi = E\Psi(\theta,\phi) \quad \cdots (3)$$

Legendrian

$$\Psi(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$$

$\Downarrow$

$$Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi) \quad \cdots (4)$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}Y + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}Y = -\frac{2IE}{\hbar^2}Y \quad \cdots (5)$$

$$\beta = \frac{2IE}{\hbar^2}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta\Phi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Theta\Phi = -\beta\Theta\Phi$$

$$\Phi\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta} + \Theta\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = -\beta\Theta\Phi \quad \ldots (6)$$

Divide eq.n. (6) thru'out by $\Theta\Phi$
and multiply by $\sin^2\theta$

$$\underbrace{\frac{1}{\Theta}\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta} + \beta\sin^2\theta}_{\text{only a function of } \theta} = \underbrace{-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}}_{\text{function of } \phi} \quad \ldots (7)$$

both sides are equal to a constant $\Rightarrow m_l^2$

$$-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = m_l^2 \quad \ldots (8)$$

$$\frac{1}{\Theta}\sin\theta\frac{\partial}{\partial\theta}\Theta + \beta\sin^2\theta = m_l^2 \quad \ldots (9)$$